

# Decomposition of Integral Self-Affine Multi-Tiles

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**Abstract.** Suppose a measurable  $\mathbb{Z}^n$ -tiling set  $K \subset \mathbb{R}^n$  is an integral self-affine multi-tile associated with an  $n \times n$  integral expansive matrix  $B$ . We provide an algorithm to decompose  $K$  into disjoint pieces  $K_j$  which satisfy  $K = \bigcup K_j$  in such a way that the collection of sets  $K_j$  is an integral self-affine collection associated with the matrix  $B$  and the number of pieces  $K_j$  is minimal. Using this algorithm, we can determine whether a given measurable  $\mathbb{Z}^n$ -tiling set  $K \subset \mathbb{R}^n$  is an integral self-affine multi-tile associated with any given  $n \times n$  integral expansive matrix  $B$ . Furthermore, we show that the minimal decomposition we provide is unique.

**Key words.** integral self-affine multi-tile, self-affine tile, integral expansive matrix

**AMS subject classifications.** 52C22

## 1 Introduction

Let  $M_n(\mathbb{Z})$  denote the set of all  $n \times n$  matrices with integer entries. A matrix  $B \in M_n(\mathbb{Z})$  is called expansive if all its eigenvalues have moduli greater than one. Equality between two measurable sets  $E, F \subseteq \mathbb{R}^n$  will always be understood modulo sets of zero measure. So  $E \cong F$  means that the symmetric difference  $(F \setminus E) \cup (E \setminus F)$  has zero Lebesgue measure. In this paper,  $|K|$  denotes the Lebesgue measure of a measurable set  $K \subset \mathbb{R}^n$ .

A finite collection of sets  $K_i \subseteq \mathbb{R}^n$ ,  $1 \leq i \leq M$ , that are compact (and possibly empty), is said to be an *integral self-affine collection* if there is an expansive matrix  $B \in M_n(\mathbb{Z})$  and finite (possibly empty) sets  $\Gamma_{ij} \subseteq \mathbb{Z}^n$ ,  $i, j = 1, \dots, M$ , such that

$$BK_i = \bigcup_{j=1}^M (\Gamma_{ij} + K_j) \text{ for } i = 1, \dots, M, \quad (1.1)$$

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and for any  $i, j, k \in \{1, \dots, M\}$

$$(\beta + K_i) \cap (\gamma + K_j) = \emptyset \text{ for } \beta \in \Gamma_{ki}, \gamma \in \Gamma_{kj} \text{ and } i \neq j \text{ or } \beta \neq \gamma. \quad (1.2)$$

The set  $\Gamma := \{\Gamma_{ij}\}_{1 \leq i, j \leq M}$  is called a *collection of digit sets* and it is called a *standard collection of digit sets* if for each  $j \in \{1, \dots, M\}$ ,  $\mathcal{D}_j := \bigcup_{i=1}^M \Gamma_{ij}$  is a complete set of coset representatives for the group  $\mathbb{Z}^n / B\mathbb{Z}^n$ . The definition above is taken from [5] (see also [6, 9]).

A finite collection of sets  $\{K_i \subseteq \mathbb{R}^n, 1 \leq i \leq M\}$  is said to  $\Lambda$ -tile  $\mathbb{R}^n$  or to be a  $\Lambda$ -tiling set (see [5]), if

$$K_i \cap K_j \cong \emptyset \text{ for } i \neq j \in \{1, \dots, M\},$$

and also, setting  $K = \bigcup_{i=1}^M K_i$ ,

$$\bigcup_{\ell \in \Lambda} (\ell + K) \cong \mathbb{R}^n,$$

and

$$(K + \ell_1) \cap (K + \ell_2) \cong \emptyset \text{ for } \ell_1, \ell_2 \in \Lambda \text{ and } \ell_1 \neq \ell_2.$$

If a collection of sets  $\{K_i \subset \mathbb{R}^n, 1 \leq i \leq M\}$  forms an integral self-affine collection and  $\Lambda$ -tiles  $\mathbb{R}^n$ , then we call  $K := \bigcup_{i=1}^M K_i$  an *integral self-affine  $\Lambda$ -tiling set with  $M$  prototiles*, or an *integral self-affine multi-tile* for short. In the particular case where  $M = 1$ , we call  $K$  an *integral self-affine tile*. It is known ([7]) that (1.1) has exactly one solution if  $M = 1$ , i.e.  $K \subset \mathbb{R}^n$  is uniquely determined by the associated  $n \times n$  expansive matrix  $B \in M_n(\mathbb{Z})$  and the digit set  $\Gamma$ . As a special class of integral self-affine multi-tiles, integral self-affine tiles have been extensively studied [4, 8, 9, 10] and they have been shown to be closely related to the theory of wavelets [6, 4, 11].

In contrast, when  $M > 1$ , there may exist several solutions of the set equation of (1.1) when some prototiles are allowed to be empty. The study of self-affine multi-tiles is not as well developed because of their complicated structure. Gröchenig and Hass [5] first provided a detailed study of integral self-affine multi-tiles (see also [12]). They constructed the general solutions of (1.1) and established the relationship between multi-wavelets and the theory of integral self-affine multi-tiles. Flaherty and Wang [1] also showed how self-affine multi-tiles can be used to construct Haar-type multiwavelets. As another application to wavelet theory, we considered the problem of constructing wavelet sets using integral self-affine multi-tiles in [3]. The wavelet sets constructed in [3] depends on

the structure of integral self-affine multi-tiles. In view of the decomposition (or the representation) of self-affine multi-tiles, an explicit form of wavelet sets can be constructed.

In the present paper, we consider the problem of representing a  $\mathbb{Z}^n$ -tiling set which is an integral self-affine multi-tile as the union of an integral self-affine collection with the minimal number of prototiles. Indeed, the representation of an integral self-affine multi-tile is not unique. For example, in dimension one, if  $K = [-\frac{1}{4}, \frac{3}{4}]$  associated with  $B = 2$ , then  $K$  is not only an integral self-affine  $\mathbb{Z}$ -tiling set with 3 prototiles, but also an integral self-affine  $\mathbb{Z}$ -tiling set with 4 prototiles (see section 3). The main goal of this paper is to provide a method to decompose an integral self-affine multi-tile  $K$  which is a  $\mathbb{Z}^n$ -tiling set into distinct almost disjoint pieces  $K_j$  which satisfy  $K = \bigcup K_j$  such that the collection of sets  $K_j$  is an integral self-affine collection associated with some expansive matrix  $B \in M_n(\mathbb{Z})$  and the number of prototiles is minimal. Moreover, the representation of integral self-affine multi-tiles will be shown to be unique.

The paper is organized as follows. In Section 2, we provide a method to represent a  $\mathbb{Z}^n$ -tiling set which is an integral self-affine multi-tile as a union of prototiles with the least number of pieces. Using this algorithm, we can determine whether or not a given measurable  $\mathbb{Z}^n$ -tiling set  $K \subset \mathbb{R}^n$  is an integral self-affine multi-tile associated with any given  $n \times n$  integral expansive matrix  $B$ . In section 3, we give some examples to illustrate our algorithm. Moreover, we construct some examples showing that some wavelet sets cannot be constructed by the method in [3] using integral self-affine multi-tiles.

## 2 The representation of integral self-affine multi-tiles

In the following, we will assume that a collection of sets  $\{K_i, 1 \leq i \leq M\}$  with  $|K_i| > 0$  for each  $i \in S := \{1, \dots, M\}$  forms an integral self-affine collection associated with an expansive matrix  $B \in M_n(\mathbb{Z})$  satisfying (1.1) and (1.2) and it  $\mathbb{Z}^n$ -tiles  $\mathbb{R}^n$ , i.e.  $K := \bigcup_{i=1}^M K_i$  is an integral self-affine  $\mathbb{Z}^n$ -tiling set with  $M$  prototiles. Moreover, we assume here that for each  $j \in S$ ,  $\mathcal{D}_j := \bigcup_{i=1}^M \Gamma_{ij}$  is a complete set of coset representatives for  $\mathbb{Z}^n / B\mathbb{Z}^n$ . This condition is known to be necessary in order for an integral self-affine multi-tile  $K$  to be a  $\mathbb{Z}^n$ -tiling set ([5]).

Define  $\Gamma_{ij}^m \subset \mathbb{Z}^n$ , for  $m \geq 1$ , by

$$B^m K_i = \bigcup_{j=1}^M (K_j + \Gamma_{ij}^m), \quad i = 1, \dots, M. \quad (2.1)$$

It follows from (1.1) that  $\Gamma_{ij}^1 = \Gamma_{ij}$ . Using (1.1) with each prototile  $K_j$ , we have

$$B^2 K_i = \bigcup_{\ell=1}^M (BK_\ell + B\Gamma_{i\ell}) = \bigcup_{\ell=1}^M \bigcup_{j=1}^M (K_j + \Gamma_{\ell j} + B\Gamma_{i\ell}) = \bigcup_{j=1}^M (K_j + \bigcup_{\ell=1}^M (\Gamma_{\ell j} + B\Gamma_{i\ell})).$$

Thus,  $\Gamma_{ij}^2 = \bigcup_{\ell=1}^M (\Gamma_{\ell j} + B\Gamma_{i\ell})$ . Inductively, we obtain

$$\Gamma_{ij}^m = \bigcup_{\ell=1}^M (\Gamma_{\ell j} + B\Gamma_{i\ell}^{m-1}). \quad (2.2)$$

Define

$$\mathcal{D}_j^m := \bigcup_{i=1}^M \Gamma_{ij}^m, \quad m \geq 1. \quad (2.3)$$

Then the corresponding self-affine multi-tile  $K = \bigcup_{i=1}^M K_i$  satisfies

$$BK = \bigcup_{i=1}^M BK_i = \bigcup_{i=1}^M \bigcup_{j=1}^M (K_j + \Gamma_{ij}) = \bigcup_{j=1}^M (K_j + \mathcal{D}_j), \quad (2.4)$$

and, more generally,

$$B^m K = \bigcup_{i=1}^M B^m K_i = \bigcup_{i=1}^M \bigcup_{j=1}^M (K_j + \Gamma_{ij}^m) = \bigcup_{j=1}^M (K_j + \mathcal{D}_j^m), \quad m \geq 1. \quad (2.5)$$

Define, for each  $m \geq 1$ , an equivalence relation  $\sim^m$  on  $S$  by

$$i \sim^m j \Leftrightarrow \mathcal{D}_i^k = \mathcal{D}_j^k, \quad 1 \leq k \leq m, \quad \text{for } i, j \in S.$$

For each  $m \geq 1$ , we will denote by  $F_1^m, \dots, F_{S_m}^m$  the corresponding equivalence classes. Obviously,  $F_1^m, \dots, F_{S_m}^m$  give a partition of  $S$ . If  $\mathcal{D}_i^m = \mathcal{D}_j^m$  for any  $m \geq 1$ , we say that  $i$  is equivalent to  $j$  and denote as  $i \sim j$ . According to this equivalence relation, we can get a partition  $\{F_j\}_{j=1}^\ell$ ,  $1 \leq \ell \leq M$ , of  $S$ , where the sets  $F_j$ ,  $1 \leq j \leq \ell$  are the corresponding equivalence classes. Then, we have the following result.

**Lemma 2.1.** Let  $K = \bigcup_{i=1}^M K_i$  be an integral self-affine  $\mathbb{Z}^n$ -tiling set with  $M$  prototiles and let  $\tilde{K}_j = \bigcup_{i \in F_j} K_i$ ,  $j = 1, \dots, \ell$ , where the partition  $\{F_j\}_{j=1}^\ell$  of  $S$  is defined as above. Then  $\tilde{K} := \bigcup_{j=1}^\ell \tilde{K}_j$  is an integral self-affine  $\mathbb{Z}^n$ -tiling set with  $\ell$  ( $\leq M$ ) prototiles.

*Proof.* Obviously,  $\tilde{K} := \bigcup_{j=1}^\ell \tilde{K}_j = \bigcup_{i=1}^M K_i$  is a  $\mathbb{Z}^n$ -tiling set by the assumption. It is left to prove that the collection  $\{\tilde{K}_j, 1 \leq j \leq \ell\}$  is an integral self-affine collection.

First, we will prove that  $\bigcup_{i \in F_s} \Gamma_{ij}$  does not depend on  $j$  for  $j \in F_t$ ,  $t \in \{1, \dots, \ell\}$ . Without loss of generality, we can assume that  $i_1 \neq i_2 \in F_t$  for some  $t \in \{1, \dots, \ell\}$ . Then for any  $m \geq 1$ , using (2.2), (2.3) and that  $S = \bigcup_{j=1}^\ell F_j$ , we have

$$\begin{aligned} \mathcal{D}_{i_1}^{m+1} = \mathcal{D}_{i_2}^{m+1} &\iff \bigcup_{i=1}^M (\Gamma_{ii_1} + B\mathcal{D}_i^m) = \bigcup_{i=1}^M (\Gamma_{ii_2} + B\mathcal{D}_i^m) \\ &\iff \bigcup_{s=1}^\ell \bigcup_{i \in F_s} (\Gamma_{ii_1} + B\mathcal{D}_i^m) = \bigcup_{s=1}^\ell \bigcup_{i \in F_s} (\Gamma_{ii_2} + B\mathcal{D}_i^m). \end{aligned} \quad (2.6)$$

Let  $\phi_{si_1} = \bigcup_{i \in F_s} \Gamma_{ii_1}$ ,  $\phi_{si_2} = \bigcup_{i \in F_s} \Gamma_{ii_2}$ . Then we have

$$\mathcal{D}_{i_1} = \bigcup_{s=1}^\ell \phi_{si_1} = \bigcup_{s=1}^\ell \phi_{si_2} = \mathcal{D}_{i_2}, \quad (2.7)$$

which is a complete set of coset representatives for  $\mathbb{Z}^n/B\mathbb{Z}^n$  by assumption. By the definition of  $F_s$ , the sets  $\mathcal{D}_i^m$  are the same for  $i \in F_s$  and any  $m \geq 1$ . We will denote this common set by  $\mathcal{D}_{\tau_s}^m$ . Then equation (2.6) can be rewritten as the following

$$\mathcal{D}_{i_1}^{m+1} = \mathcal{D}_{i_2}^{m+1} \iff \bigcup_{s=1}^\ell (\phi_{si_1} + B\mathcal{D}_{\tau_s}^m) = \bigcup_{s=1}^\ell (\phi_{si_2} + B\mathcal{D}_{\tau_s}^m). \quad (2.8)$$

Suppose that, for some  $s \in \{1, \dots, \ell\}$ ,  $\phi_{si_1} \neq \phi_{si_2}$ . Then  $\phi_{si_1} \setminus \phi_{si_2} \neq \emptyset$  or  $\phi_{si_2} \setminus \phi_{si_1} \neq \emptyset$ .

WLOG, we assume that there exists  $x \in \phi_{si_1} \setminus \phi_{si_2}$ , then  $x \in \phi_{ti_2}$  for some  $t \in \{1, 2, \dots, \ell\}$  and  $t \neq s$  by (2.7). Next, we will prove that  $(x + B\mathcal{D}_{\tau_s}^m) \cap (y + B\mathcal{D}_{\tau_w}^m) = \emptyset$  for any  $y \in \phi_{wi_2}$ ,  $w \in \{1, 2, \dots, \ell\}$  and  $y \neq x$ . Otherwise, there is  $y \in \phi_{wi_2}$  and  $y \neq x$  such that  $(x + B\mathcal{D}_{\tau_s}^m) \cap (y + B\mathcal{D}_{\tau_w}^m) \neq \emptyset$ . This implies that  $(x - y) \in B\mathbb{Z}^n$ , which gives a contradiction since  $x \in \phi_{si_1} \subset \mathcal{D}_{i_1}$ ,  $y \in \phi_{wi_2} \subset \mathcal{D}_{i_2}$  and  $\mathcal{D}_{i_1} = \mathcal{D}_{i_2}$  is a complete set of coset representatives for  $\mathbb{Z}^n/B\mathbb{Z}^n$  by (2.7). Hence, it follows from (2.8) that the only possibility is  $x + B\mathcal{D}_{\tau_s}^m = x + B\mathcal{D}_{\tau_t}^m$ , which forces that  $\mathcal{D}_{\tau_s}^m = \mathcal{D}_{\tau_t}^m$  for any  $m \in \mathbb{N}$ ,

contradicting the fact that  $F_s$  and  $F_t$  are different equivalence classes. Therefore,  $\phi_{si_1} = \phi_{si_2}$  for any  $i_1, i_2 \in F_t$ . Let  $\Lambda_{st} = \bigcup_{i \in F_s} \Gamma_{ij}$ ,  $j \in F_t$ . Then

$$B\tilde{K}_s = B \bigcup_{i \in F_s} K_i = \bigcup_{i \in F_s} \bigcup_{j=1}^M (K_j + \Gamma_{ij}) = \bigcup_{t=1}^\ell \bigcup_{j \in F_t} (K_j + \bigcup_{i \in F_s} \Gamma_{ij}) = \bigcup_{t=1}^\ell (\tilde{K}_t + \Lambda_{st}).$$

This proves that  $\{\tilde{K}_j, 1 \leq j \leq \ell\}$  is an integral self-affine collection.  $\square$

Lemma 2.1 provides us an idea to decompose  $K$  according to the equivalence relation “ $\sim$ ”. Combining (2.5), we can decompose it by finding the non-empty intersection of different integer translations between its  $B$ -dilations and itself.

For  $m \geq 1$ , let  $C_m$  be the collection of non-empty sets of the form

$$(B^{k_1}K + \ell_1) \cap (B^{k_2}K + \ell_2) \cap \cdots \cap (B^{k_r}K + \ell_r) \cap K,$$

where  $1 \leq k_i \leq m$ ,  $\ell_i \in \mathbb{Z}^n$ ,  $r \geq 1$ . Then  $C_1 \subseteq C_2 \subseteq \cdots$  and each  $C_m$  is finite since, for a fixed  $k \geq 1$ , only finitely integral translates of  $B^k K$  can intersect  $K$ . Furthermore,  $C_m$  is stable under intersection, i.e.  $A, B \in C_m$  implies that  $A \cap B \in C_m$  if  $A \cap B \neq \emptyset$ .

For each  $m \geq 1$ , we order the sets in  $C_m$  by inclusion and we let  $C'_m$  be the collection of all minimal sets in  $C_m$  (i.e. if  $A_1 \subset A_2 \subset \cdots$ , then the minimal set of sequence  $A_j$  is  $A_1$ ).  $C'_m$  is the collection of such minimal sets picked in each inclusion sequence in  $C_m$ ). Then the elements of  $C'_m$  are of the form

$$\bigcap_{\ell_m \in L_m} \cdots \bigcap_{\ell_1 \in L_1} (B^m K + \ell_m) \cap \cdots \cap (BK + \ell_1) \cap K \quad (2.9)$$

for some subsets  $L_i \subset \mathbb{Z}^n$ ,  $i = 1, \dots, m$ , and any element of  $C_m$  can be written as a union of sets in  $C'_m$ . Clearly, the collection of sets in  $C'_m$  forms a partition of  $K$ .

**Lemma 2.2.** *Let  $K$  be an integral self-affine  $\mathbb{Z}^n$ -tiling set with  $M$  prototiles. For each  $m \geq 1$ , a set  $E \in C'_m$  if and only if  $E = \bigcup_{i \in F_s^m} K_i$ , for some  $s$  with  $1 \leq s \leq S_m$ .*

*Proof.* We will first prove that if  $i_1 \in S$ ,  $K_{i_1} \subset E \in C'_m$  and  $i_2 \stackrel{m}{\sim} i_1$ , then  $K_{i_2} \subset E$ . Indeed, any

$E \in C'_m$  can be written as

$$\begin{aligned}
E &= \bigcap_{\ell_m \in L_m} \cdots \bigcap_{\ell_1 \in L_1} (B^m K + \ell_m) \cap \cdots \cap (BK + \ell_1) \cap K \\
&= \bigcap_{\ell_m \in L_m} \cdots \bigcap_{\ell_1 \in L_1} \bigcup_{j_0, j_1, \dots, j_m=1}^M (K_{j_m} + \mathcal{D}_{j_m}^m + \ell_m) \cap \cdots \cap (K_{j_1} + \mathcal{D}_{j_1} + \ell_1) \cap K_{j_0} \\
&= \bigcap_{\ell_m \in L_m} \cdots \bigcap_{\ell_1 \in L_1} \bigcup_{j=1}^M (K_j + \mathcal{D}_j^m + \ell_m) \cap \cdots \cap (K_j + \mathcal{D}_j + \ell_1) \cap K_j.
\end{aligned}$$

It follows that the inclusion  $K_{i_1} \subset E$  is equivalent to  $L_1 \subseteq -\mathcal{D}_{i_1}^1, L_2 \subseteq -\mathcal{D}_{i_1}^2, \dots, L_m \subseteq -\mathcal{D}_{i_1}^m$ .

Thus, if  $i_2 \stackrel{m}{\sim} i_1$ , we have  $K_{i_2} \subset E \Leftrightarrow K_{i_1} \subset E$ .

Using the definition of  $C_m$  and the fact that  $K$  is an integral self-affine  $\mathbb{Z}^n$ -tiling set, it is easy to see that any element of  $C_m$ , and thus of  $C'_m$ , is the union of some sets  $K_i$ ,  $i \in I \subseteq S$ . To prove the converse, we will use induction on  $m$ . For  $m = 1$ , suppose that  $\bigcap_{\ell_1 \in L_1} (BK + \ell_1) \cap K \in C'_1$ . Then there exists some  $T_1 \subseteq S$  such that

$$\bigcap_{\ell_1 \in L_1} (BK + \ell_1) \cap K = \bigcap_{\ell_1 \in L_1} \bigcup_{j=1}^M (K_j + \mathcal{D}_j + \ell_1) \cap K_j = \bigcup_{i \in T_1} K_i,$$

which implies that  $L_1 \subseteq \bigcap_{i \in T_1 \subseteq S} -\mathcal{D}_i$ . If  $T_1$  contains at least two different elements, then  $\mathcal{D}_{i_1} = \mathcal{D}_{i_2}$  for any  $i_1, i_2 \in T_1$ . Otherwise, we can find  $p \in -(\mathcal{D}_{i_1} \setminus \mathcal{D}_{i_2})$  and we have

$$K_{i_1} \subseteq (BK + p) \cap K \in C_1 \text{ and } K_{i_2} \cap (BK + p) \cap K = \emptyset.$$

Thus, we obtain that

$$\emptyset \neq K_{i_1} \subseteq \bigcap_{\ell_1 \in L_1} (BK + \ell_1) \cap K \cap (BK + p) = \bigcup_{i \in T_1} K_i \cap (BK + p) \cap K \subseteq \bigcup_{i \in T_1 \setminus \{i_2\}} K_i,$$

which contradicts the fact that  $\bigcap_{\ell_1 \in L_1} (BK + \ell_1) \cap K = \bigcup_{i \in T_1} K_i$  is the minimal set. This proves our claim for  $m = 1$ . Suppose that our claim is true for some  $m > 1$ . Then we will prove it is true for  $m + 1$ . Let

$$\bigcap_{\ell_{m+1} \in L_{m+1}} \cdots \bigcap_{\ell_1 \in L_1} (B^{m+1} K + \ell_{m+1}) \cap \cdots \cap (BK + \ell_1) \cap K = \bigcup_{i \in T_{m+1} \subseteq S} K_i \in C'_{m+1}.$$

Since any element of  $C'_{m+1}$  is contained in some element of  $C'_m$ , we here have  $\bigcup_{i \in T_{m+1}} K_i \subseteq \bigcup_{i \in T_m} K_i \in C'_m$  for some set  $T_m \subset S$ . Next we will prove that  $T_{m+1}$  is an equivalence class associated with  $\stackrel{m+1}{\sim}$ . Using our inclusion hypothesis,  $T_m$  is an equivalence class associated with  $\stackrel{m}{\sim}$ . If  $T_{m+1}$  contains only

one element, we are done. Assume that  $T_{m+1}$  contains at least two different elements, say  $i_1 \neq i_2$ , and  $\mathcal{D}_{i_1}^{m+1} \neq \mathcal{D}_{i_2}^{m+1}$ . Then there exists  $p \in -(\mathcal{D}_{i_1}^{m+1} \setminus \mathcal{D}_{i_2}^{m+1})$  such that

$$K_{i_1} \subseteq (B^{m+1}K + p) \cap K \in C_{m+1} \text{ and } K_{i_2} \cap (B^{m+1}K + p) \cap K = \emptyset.$$

Hence, we have

$$\begin{aligned} K_{i_1} &\subseteq \bigcap_{\ell_{m+1} \in L_{m+1}} \cdots \bigcap_{\ell_1 \in L_1} (B^{m+1}K + \ell_{m+1}) \cap \cdots \cap (BK + \ell_1) \cap K \cap (B^{m+1}K + p) \\ &\subseteq \bigcup_{i \in T_{m+1} \setminus \{i_2\}} K_i, \end{aligned}$$

which implies that  $\bigcup_{i \in T_{m+1}} K_i$  is not the minimal one. This is a contradiction. So  $\mathcal{D}_{i_1}^{m+1} = \mathcal{D}_{i_2}^{m+1}$ .

On the other hand, since  $i_1 \stackrel{m}{\sim} i_2$  for any  $i_1, i_2 \in T_{m+1} \subseteq T_m$ , we have  $\mathcal{D}_{i_1}^k = \mathcal{D}_{i_2}^k$  for  $k = 1, \dots, m$ . This proves that  $T_{m+1}$  is an equivalence class for the equivalence under  $\stackrel{m+1}{\sim}$ .  $\square$

As we mentioned in the introduction, the representation of an integral self-affine  $\mathbb{Z}^n$ -tiling set is not unique. For convenience, we call the representation of an integral self-affine tiling set  $K = \bigcup_{i=1}^M K_i$  to be *in its simplest form* if the number  $M$  is the minimal.

Given an integral self-affine  $\mathbb{Z}^n$ -tiling set  $K$ , the sets in  $C'_m$  form a partition of  $K$ . Let  $C'_m = \{K_{mi}, i = 1, \dots, S_m\}$ . Since each set in  $C_m$  is a finite union of the prototiles making up the set  $K$ , the cardinality of  $C_m$  is bounded independently of  $m$  and thus, for some  $m_0 \geq 1$ ,  $C_m = C_{m_0}$  for  $m \geq m_0$ , and thus  $C'_m = C'_{m_0}$  for  $m \geq m_0$ . We let  $S_{m_0} = N$  and  $K_{m_0 i} = W_i$ ,  $i = 1, \dots, N$ . The procedure just described provides an algorithm to decompose any integral self-affine  $\mathbb{Z}^n$ -tiling set into disjoint prototiles with a representation in its simplest form, as shown in the following theorem.

**Theorem 2.3.** *Suppose that  $K$  is an integral self-affine  $\mathbb{Z}^n$ -tiling set. Then  $K = \bigcup_{i=1}^N W_i$  is an integral self-affine  $\mathbb{Z}^n$ -tiling set with  $N$  prototiles, where  $W_i$ ,  $i = 1, \dots, N$  are defined as above. Furthermore, the representation  $K = \bigcup_{i=1}^N W_i$  is in its simplest form and the simplest form representation of  $K$  is unique.*

*Proof.* Since  $K$  is an integral self-affine  $\mathbb{Z}^n$ -tiling set, there exists a partition  $\{K_i\}_{i=1}^M$  of  $K$  such that  $K = \bigcup_{i=1}^M K_i$  is an integral self-affine  $\mathbb{Z}^n$ -tiling set with  $M$  prototiles. By the definition of  $W_i$ , the sets  $W_i$ ,  $i = 1, \dots, N$  are essentially disjoint and Lemma 2.2 implies that  $W_i = \bigcup_{j \in T_i \subseteq S} K_j$



for each  $i = 1, \dots, N$ , where the sets  $T_i, i = 1, \dots, N$  are the equivalence classes obtained by the equivalence relation “ $\sim$ ”. Thus  $S = \bigcup_{i=1}^N T_i$ . It follows from Lemma 2.1 that the collection  $\{W_i\}_{i=1}^N$  is an integral self-affine collection. Therefore,  $K = \bigcup_{i=1}^N W_i$  is an integral self-affine  $\mathbb{Z}^n$ -tiling set with  $N$  prototiles. We note that the sets  $W_i, i = 1, \dots, N$  do not depend on the sets  $K_i, i = 1, \dots, M$ . For any representation of the set  $K$  which satisfies the integral self-affine conditions, the set  $W_i$  is the union of some prototiles for each  $i = 1, \dots, N$ . Thus, we have  $N \leq M$ . This proves that the representation  $K = \bigcup_{i=1}^N W_i$  is in its simplest form and the simplest form representation of  $K$  is unique.  $\square$

Theorem 2.3 shows that if an integral self-affine  $\mathbb{Z}^n$ -tiling set  $K = \bigcup_{i=1}^M K_i$  is not in its simplest form, then there exists a partition  $\{F_j\}_{j=1}^\ell$  of the set  $S$  with  $F_{j_0}$  having at least two elements for at least one  $j_0$ , such that the collection  $\{\bigcup_{i \in F_j} K_i\}_{j=1}^\ell$  is an integral self-affine collection. Combining all above results, we get the following corollary, which provides an equivalent condition for an integral self-affine  $\mathbb{Z}^n$ -tiling set to be in its simplest form.

**Corollary 2.4.** *Let  $K = \bigcup_{i=1}^M K_i$  be an integral self-affine  $\mathbb{Z}^n$ -tiling set with  $M$  prototiles. Then, the representation  $K = \bigcup_{i=1}^M K_i$  is in its simplest form if and only if for any  $i_1, i_2 \in S$  with  $i_1 \neq i_2$ , there exists some  $m \geq 1$  such that  $\mathcal{D}_{i_1}^m \neq \mathcal{D}_{i_2}^m$ .*

Theorem 2.3 provides us an algorithm for decomposing an integral self-affine  $\mathbb{Z}^n$ -tiling set  $K$  into essentially disjoint prototiles  $K_j, j = 1, \dots, M$  such that the collection  $\{K_j\}_{j=1}^M$  is an integral self-affine collection. Moreover, this representation is in its simplest form and the decomposition is unique in the sense that the number of prototiles is minimal by Corollary 2.4 and also, in the sense, that given any representation of  $K$  as a union of prototiles  $K_j, j = 1, \dots, M$ , the elements of the minimal representation can always be written as finite unions of these sets  $K_j$ . Given an integral self-affine  $\mathbb{Z}^n$ -tiling set  $K$ , we compute the collection  $C'_m$  step by step until we find an integer  $m_0$  such that  $C'_m = C'_{m_0}$  for any  $m \geq m_0$ . Or, alternatively, we check whether or not the collection  $C'_m$  obtained at each step is an integral self-affine collection. If it is, we stop and  $K = \bigcup_{i=1}^{S_m} K_{mi}$  is an integral self-affine  $\mathbb{Z}^n$ -tiling set and the representation is in its simplest form.

As we mentioned before, there are many integral self-affine  $\mathbb{Z}^n$ -tiling sets which have different representations. However, the representation we provide here is unique by Corollary 2.4. Such

examples will be given in section 3. Furthermore, we can also use this algorithm to determine whether or not a given measurable  $\mathbb{Z}^n$ -tiling set  $K \subset \mathbb{R}^n$  is an integral self-affine multi-tile associated with any given  $n \times n$  integral expansive matrix  $B$ . We use the above algorithm to decompose the set  $K$ . Then  $K$  is an integral self-affine  $\mathbb{Z}^n$ -tiling set if and only if the process stops after finitely many steps.

Furthermore, our algorithm can only be applied to a measurable  $\mathbb{Z}^n$ -tiling set to determine if it is an integral self-affine multi-tile. For a non  $\mathbb{Z}^n$ -tiling set, our algorithm is not applicable.

### 3 Examples

For some integral self-affine tiling sets with simple geometrical shape, it is easy to see how to decompose the given measurable set  $K \subset \mathbb{R}^n$  into almost disjoint pieces  $K_j$  such that  $K_j$ ,  $j \in I$ , where  $I$  is a finite set, is an integral self-affine collection. However, for those with complicated geometrical shape, it might not be easy to represent it as an integral self-affine collection. For such self-affine tiling sets, we can use the method introduced in section 2 to solve this problem. In this section, we will give some examples to show how to use the algorithm given in section 2 to represent an integral self-affine  $\mathbb{Z}^n$ -tiling set in its simplest form.

**Example 3.1.** *In dimension one, consider the set  $K = [-\frac{3}{4}, \frac{1}{4}]$  associated with  $B = 2$ .*

The set  $K$  here can be not only an integral self-affine  $\mathbb{Z}$ -tiling set with 4 prototiles, but an integral self-affine  $\mathbb{Z}$ -tiling set with 3 prototiles. In the following, we will give its representation for each case. For the first case, let

$$K_1 = [-\frac{3}{4}, -\frac{1}{2}], K_2 = [-\frac{1}{2}, -\frac{1}{4}], K_3 = [-\frac{1}{4}, 0], K_4 = [0, \frac{1}{4}].$$

Then, we have

$$\begin{aligned} BK_1 &= [-\frac{3}{2}, -1] = (K_2 - 1) \bigcup (K_3 - 1) \implies \Gamma_{11} = \emptyset, \Gamma_{12} = \{-1\}, \Gamma_{13} = \{-1\}, \Gamma_{14} = \emptyset, \\ BK_2 &= [-1, -\frac{1}{2}] = (K_4 - 1) \bigcup K_1 \implies \Gamma_{21} = \{0\}, \Gamma_{22} = \emptyset, \Gamma_{23} = \emptyset, \Gamma_{24} = \{-1\}, \\ BK_3 &= [-\frac{1}{2}, 0] = K_2 \bigcup K_3 \implies \Gamma_{31} = \emptyset, \Gamma_{32} = \{0\}, \Gamma_{33} = \{0\}, \Gamma_{34} = \emptyset, \\ BK_4 &= [0, \frac{1}{2}] = (K_1 + 1) \bigcup K_4 \implies \Gamma_{41} = \{1\}, \Gamma_{42} = \emptyset, \Gamma_{43} = \emptyset, \Gamma_{44} = \{0\}, \end{aligned}$$

and

$$\begin{aligned}\mathcal{D}_1 &= \bigcup_{i=1}^4 \Gamma_{i1} = \{0, 1\}, \quad \mathcal{D}_2 = \bigcup_{i=1}^4 \Gamma_{i2} = \{-1, 0\}, \\ \mathcal{D}_3 &= \bigcup_{i=1}^4 \Gamma_{i3} = \{-1, 0\}, \quad \mathcal{D}_4 = \bigcup_{i=1}^4 \Gamma_{i4} = \{-1, 0\}.\end{aligned}$$

This shows that for each  $j \in \{1, 2, 3, 4\}$ ,  $\mathcal{D}_j$  is a complete set of coset representatives for the group  $\mathbb{Z}/2\mathbb{Z}$  and the set  $K$  is an integral self-affine  $\mathbb{Z}$ -tiling set with 4 prototiles. Define  $\mathcal{D}_j^m$  as in (2.3). It follows from (2.2) and (2.3) that  $\mathcal{D}_j^m = \bigcup_{i=1}^4 (\Gamma_{ij} + 2\mathcal{D}_i^{m-1})$ . Note that for  $i \in \{1, 2, 3, 4\}$ ,  $\Gamma_{i2} = \Gamma_{i3}$ . Thus we get  $\mathcal{D}_2^m = \mathcal{D}_3^m$  for any  $m \geq 1$  by the definition of  $\mathcal{D}_j^m$ . On the other hand, since

$$\mathcal{D}_1^2 = \bigcup_{i=1}^4 (\Gamma_{i1} + 2\mathcal{D}_i) = \{-2, -1, 0, 1\} = \mathcal{D}_2^2, \quad \mathcal{D}_4^2 = \bigcup_{i=1}^4 (\Gamma_{i4} + 2\mathcal{D}_i) = \{-3, -2, -1, 0\},$$

it follows that  $\mathcal{D}_1 \neq \mathcal{D}_2$ ,  $\mathcal{D}_1^2 \neq \mathcal{D}_4^2$  and  $\mathcal{D}_2^2 \neq \mathcal{D}_4^2$ . The equivalence classes for the equivalence relation  $\sim$  are thus  $\{1\}$ ,  $\{2, 3\}$  and  $\{4\}$ . By Theorem 2.3,  $K = \bigcup_{i=1}^4 K_i$  is not in “the simplest form”. By the proof in Theorem 2.3, we let

$$K'_1 = [-\frac{3}{4}, -\frac{1}{2}], \quad K'_2 = [-\frac{1}{2}, 0], \quad K'_3 = [0, \frac{1}{4}].$$

Define  $\Gamma'_{ij}$ ,  $i, j = 1, 2, 3$  to satisfy  $BK'_i = \bigcup_{j=1}^3 (K'_j + \Gamma'_{ij})$  and  $\mathcal{D}'_j = \bigcup_{i=1}^3 \Gamma'_{ij}$ . Then, we have

$$\begin{aligned}BK'_1 &= [-\frac{3}{2}, -1] = (K'_2 - 1) \implies \Gamma'_{11} = \emptyset, \quad \Gamma'_{12} = -1, \quad \Gamma'_{13} = \emptyset, \\ BK'_2 &= [-1, 0] = K'_1 \bigcup K'_2 \bigcup (K'_3 - 1) \implies \Gamma'_{21} = 0, \quad \Gamma'_{22} = 0, \quad \Gamma'_{23} = -1, \\ BK'_3 &= [0, \frac{1}{2}] = (K'_1 + 1) \bigcup K'_3 \implies \Gamma'_{31} = 1, \quad \Gamma'_{32} = \emptyset, \quad \Gamma'_{33} = 0.\end{aligned}$$

Furthermore,

$$\mathcal{D}'_1 = \bigcup_{i=1}^3 \Gamma'_{i1} = \{0, 1\}, \quad \mathcal{D}'_2 = \bigcup_{i=1}^3 \Gamma'_{i2} = \{-1, 0\}, \quad \mathcal{D}'_3 = \bigcup_{i=1}^3 \Gamma'_{i3} = \{-1, 0\}.$$

This shows that for each  $i \in \{1, 2, 3\}$ ,  $\mathcal{D}'_i$  is a complete set of coset representatives for  $\mathbb{Z}/2\mathbb{Z}$  and that  $K = \bigcup_{i=1}^3 K'_i$  is an integral self-affine  $\mathbb{Z}$ -tiling set with 3 prototiles.  $\square$

*Remark 3.1.* Let  $K = \bigcup_{j=1}^M K_j$  be an integral self-affine  $\mathbb{Z}^n$ -tiling set with  $M$  prototiles. If  $\Gamma_{ij_1} = \Gamma_{ij_2}$  for any  $i \in S$ , then  $j_1 \sim j_2$ . But the converse is not necessarily true as shown in the next example.

**Example 3.2.** In dimension one, consider the set  $K = [-\frac{3}{4}, \frac{1}{4}]$  associated with  $B = -3$ .

Obviously,  $K$  is a  $\mathbb{Z}$ -tiling set. For this example, the set  $K$  can be represented as a union of many different kinds of prototiles. The simplest form is the representation of  $K$  as an integral self-affine tile, since

$$BK = [-\frac{3}{4}, \frac{9}{4}] = K \bigcup (K+1) \bigcup (K+2).$$

On the other hand, we can also let  $K_1 = [-\frac{3}{4}, -\frac{1}{4}]$  and  $K_2 = [-\frac{1}{4}, \frac{1}{4}]$ , then

$$\begin{aligned} BK_1 &= [-\frac{3}{4}, \frac{9}{4}] = (K_1 + 2) \bigcup (K_2 + \{1, 2\}) \implies \Gamma_{11} = \{2\}, \Gamma_{12} = \{1, 2\}, \\ BK_2 &= [-\frac{3}{4}, \frac{3}{4}] = (K_1 + \{0, 1\}) \bigcup K_2 \implies \Gamma_{21} = \{0, 1\}, \Gamma_{22} = \{0\}, \end{aligned}$$

and

$$\mathcal{D}_1 = \bigcup_{i=1}^2 \Gamma_{i1} = \{0, 1, 2\} = \bigcup_{i=1}^2 \Gamma_{i2} = \mathcal{D}_2.$$

This shows that  $K = \bigcup_{i=1}^2 K_i$  is an integral self-affine  $\mathbb{Z}$ -tiling set with 2 prototiles. Hence,  $K$  is not only an integral self-affine  $\mathbb{Z}$ -tiling set with one prototiles, but an integral self-affine  $\mathbb{Z}$ -tiling set with two prototiles. Corollary 2.4 shows that  $\mathcal{D}_1^m = \mathcal{D}_2^m$  for any  $m \geq 1$ , i.e.  $1 \sim 2$ . However,  $\Gamma_{i1} \neq \Gamma_{i2}$  for any  $i = 1, 2$ .  $\square$

**Example 3.3.** In dimension two, consider the set  $K = H \bigcup (-H) \bigcup K'$  associated with the matrix  $B = \begin{pmatrix} -1 & 1 \\ -3 & 1 \end{pmatrix}$ , where

$$H = \text{conv}\left\{\begin{pmatrix} 1/3 \\ 1 \end{pmatrix}, \begin{pmatrix} 2/3 \\ 1 \end{pmatrix}, \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix}, \begin{pmatrix} 1/6 \\ 1/2 \end{pmatrix}\right\} \text{ and } K' = \text{conv}\left\{\begin{pmatrix} -1/6 \\ 1/2 \end{pmatrix}, \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix}, \begin{pmatrix} 1/6 \\ -1/2 \end{pmatrix}, \begin{pmatrix} -1/2 \\ -1/2 \end{pmatrix}\right\},$$

where  $\text{conv}(E)$  denotes the convex hull of  $E$ .

It is easy to see that  $K$  is a  $\mathbb{Z}^2$ -tiling set. The sets  $K$  and  $BK$  are depicted in Figure 1. Clearly, we can divide  $K$  into six pieces  $\{K_j\}_{j=1}^6$  with  $K_1 = H, K_2 = E, K_3 = F, K_4 = -E, K_5 = -F, K_6 = -H$ , where

$$E = \text{conv}\left\{\begin{pmatrix} 1/3 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix}, \begin{pmatrix} 1/6 \\ 1/2 \end{pmatrix}\right\} \text{ and } F = \text{conv}\left\{\begin{pmatrix} -1/6 \\ 1/2 \end{pmatrix}, \begin{pmatrix} 1/6 \\ 1/2 \end{pmatrix}, \begin{pmatrix} -1/3 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}\right\}.$$

Moreover, we have (see Figure 1)

$$BK_1 = (K_1 + \begin{pmatrix} 0 \\ -1 \end{pmatrix}) \cup (K_2 + \begin{pmatrix} 0 \\ -1 \end{pmatrix}), \quad BK_2 = (K_3 + \begin{pmatrix} 0 \\ -1 \end{pmatrix}) \cup K_5, \quad BK_3 = K_1 \cup K_2, \\ BK_4 = K_3 \cup (K_5 + \begin{pmatrix} 0 \\ 1 \end{pmatrix}), \quad BK_5 = K_4 \cup K_6, \quad BK_6 = (K_4 + \begin{pmatrix} 0 \\ 1 \end{pmatrix}) \cup (K_6 + \begin{pmatrix} 0 \\ 1 \end{pmatrix}),$$

which implies that  $\{K_j\}_{j=1}^6$  is an integral self-affine collection. Therefore,  $K$  is an integral self-affine  $\mathbb{Z}^n$ -tiling set with 6 prototiles. However, this representation is not in its simplest form. We will use the algorithm introduced in section 2 to represent the set  $K$  in its simplest form. At the first step, we get a partition  $C'_1 = \{K_{1i}\}_{i=1}^2$  of  $K$  by computing (2.9) for  $m = 1$  (see Figure 2).

$$K_{11} = (BK + \begin{pmatrix} 0 \\ 1 \end{pmatrix}) \cap K = K_1 \cup K_2 \cup K_3, \quad K_{12} = (BK + \begin{pmatrix} 0 \\ -1 \end{pmatrix}) \cap K = K_4 \cup K_5 \cup K_6.$$

It is easy to check that  $\{K_{1i}\}_{i=1}^2$  is not an integral self-affine collection. Thus, we need to decompose  $K_{1i}$ ,  $i = 1, 2$  further using (2.9) (see Figure 3) and we have

$$K_{21} = (B^2K + \begin{pmatrix} 1 \\ 1 \end{pmatrix}) \cap K_{11} = K_1 \cup K_2, \quad K_{22} = (B^2K + \begin{pmatrix} -1 \\ -1 \end{pmatrix}) \cap K_{11} = K_3, \\ K_{23} = (B^2K + \begin{pmatrix} -1 \\ -1 \end{pmatrix}) \cap K_{12} = K_4 \cup K_6, \quad K_{24} = (B^2K + \begin{pmatrix} 1 \\ 1 \end{pmatrix}) \cap K_{12} = K_5.$$

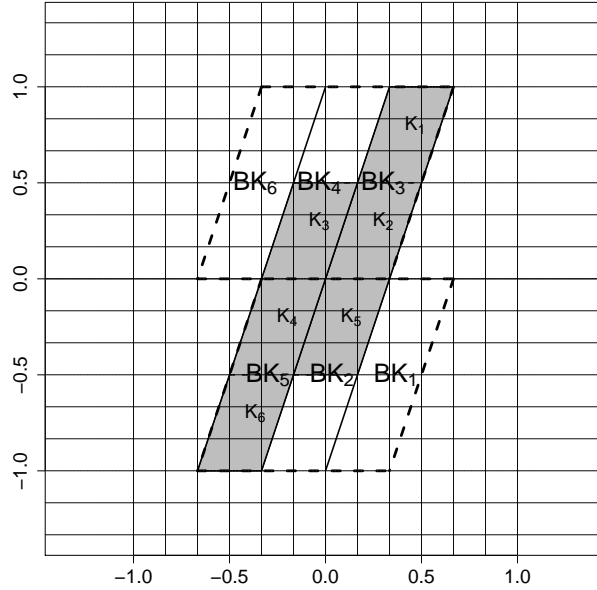
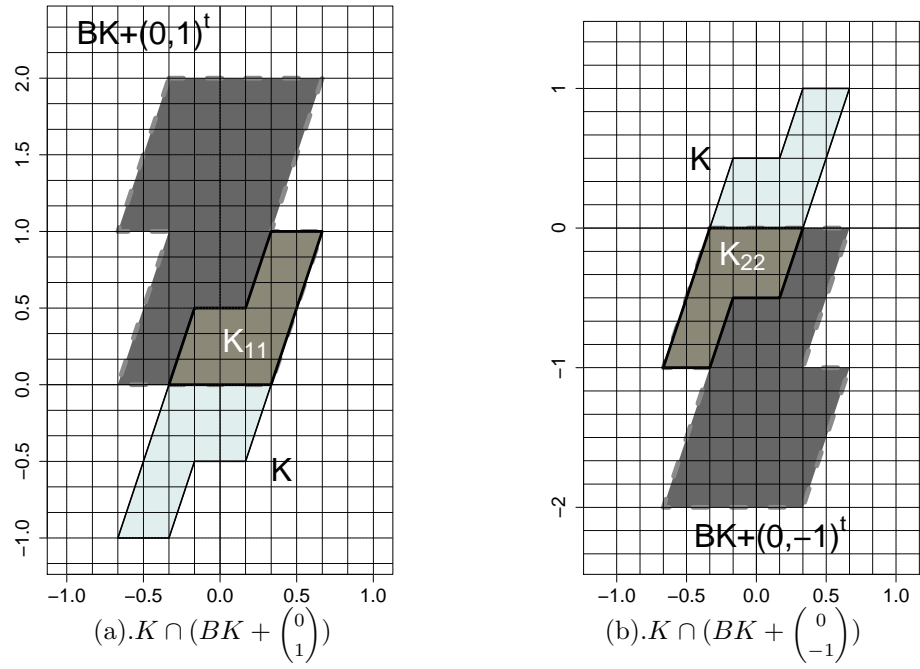


Figure 1:  $K_i$  and its  $B$ -dilation  $BK_i$ ,  $i=1,2,3,4$

Figure 2: The intersection of  $K$  and integer translations of  $BK$

Furthermore,  $\{K_{2i}\}_{i=1}^4$  is an integral self-affine collection since

$$\begin{aligned} BK_{21} &= (K_{21} + \begin{pmatrix} 0 \\ -1 \end{pmatrix}) \cup (K_{21} + \begin{pmatrix} 0 \\ -1 \end{pmatrix}) \cup K_{24}, \quad BK_{22} = K_{21}, \\ BK_{23} &= (K_{23} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}) \cup K_{22} \cup (K_{24} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}), \quad BK_{24} = K_{23}. \end{aligned}$$

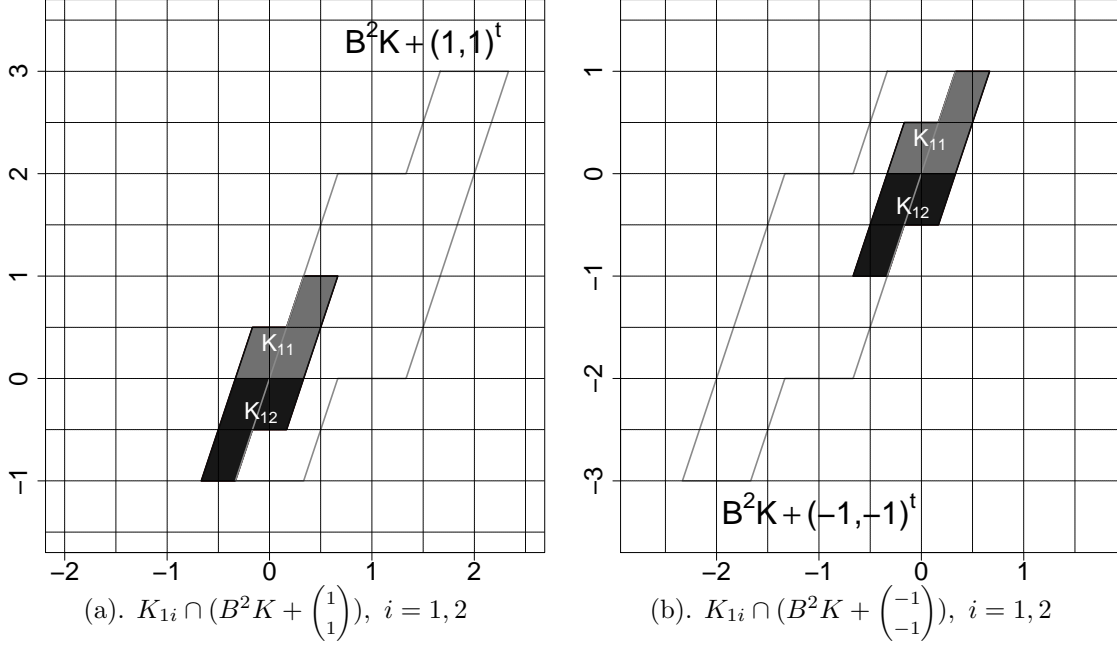


Figure 3: The intersection of  $K_{1i}$ ,  $i = 1, 2$  and integer translations of  $B^2K$

Therefore,  $K = \bigcup_{i=1}^4 K_{2i}$  is an integral self-affine  $\mathbb{Z}^2$ -tiling set with 4 prototiles and this representation is in its simplest form.  $\square$

It has been shown in [5] that the theory of integral self-affine multi-tiles is closely related to the theory of wavelets. We also considered in [3] the problem of constructing wavelet sets using integral self-affine multi-tiles and gave a sufficient condition for an integral self-affine  $\mathbb{Z}^n$ -tiling set with multi-prototiles to be a scaling set. The example below shows that some wavelet sets cannot be constructed using integral self-affine  $\mathbb{Z}^n$ -tiling sets with multi-prototiles as was done in [3].

**Example 3.4.** In dimension one, consider the set  $K = [-a, 1-a]$  where  $0 < a < 1$  associated with  $B = 2$ . Then  $K$  is an integral self-affine multi-tile if and only if  $a \in \mathbb{Q}$ .

*Proof.* It has been proved in [2] that  $K$  is a 2-dilation MRA scaling set and the set  $Q := 2K \setminus K$  is a 2-dilation MRA wavelet set. Obviously,  $K$  is a  $\mathbb{Z}$ -tiling set. We will divide into two cases to prove our claim.

Case 1:  $a \in \mathbb{Q}$ . Then  $a = \frac{p}{q}$ , for some  $p, q \in \mathbb{N}$  and  $(p, q) = 1$ ,  $p < q$  since  $0 < a < 1$ . In this case,  $K = [-\frac{p}{q}, 1 - \frac{p}{q}] = [-\frac{p}{q}, \frac{q-p}{q}]$ . Let

$$K_i = [-\frac{p-i+1}{q}, -\frac{p-i}{q}], \quad i = 1, \dots, q.$$

Then  $K = \bigcup_{i=1}^q K_i$ , where the union is essentially disjoint and we have

$$\begin{aligned} BK_i &= 2[-\frac{p-i+1}{q}, -\frac{p-i}{q}] = [-\frac{2p-2i+2}{q}, -\frac{2p-2i}{q}] \\ &= [-\frac{2(p-i+1)}{q}, -\frac{2p-2i+1}{q}] \cup [-\frac{2p-2i+1}{q}, -\frac{2(p-i)}{q}]. \end{aligned}$$

Note that

$$-\frac{2(p-i+1)}{q} \in \left\{ -\frac{p}{q}, -\frac{p-1}{q}, \dots, -\frac{1}{q}, 0, \frac{1}{q}, \dots, \frac{q-p-1}{q} \right\} + \mathbb{Z},$$

and

$$-\frac{2(p-i)}{q} \in \left\{ -\frac{p-1}{q}, -\frac{p-2}{q}, \dots, 0, \frac{1}{q}, \dots, \frac{q-p}{q} \right\} + \mathbb{Z}.$$

Thus, we have  $BK_i = (K_{j_1} + \ell_1) \cup (K_{j_2} + \ell_2)$  for some  $\ell_1, \ell_2 \in \mathbb{Z}$ , and  $j_1, j_2 \in \{1, 2, \dots, q\}$ . This proves that the collection  $\{K_i\}_{i=1}^q$  is an integral self-affine collection. Therefore,  $K = \bigcup_{i=1}^q K_i$  is an integral self-affine  $\mathbb{Z}$ -tiling set with  $q$  prototiles.

Case 2:  $a \notin \mathbb{Q}$ . Assume that  $K = [-a, 1-a]$  is an integral self-affine  $\mathbb{Z}$ -tiling set with  $M$  prototiles. Then we can use the method given in section 2 to decompose  $K$  as essentially disjoint union of prototiles. First, we compute  $C'_1$  using (2.9) and get a partition  $\{K_{1i}\}_{i=1}^2$  of  $K$  which are defined by

$$K_{11} := (BK + 1) \cap K = [1 - 2a, 3 - 2a] \cap [-a, 1 - a] = [1 - 2a, 1 - a],$$

$$K_{12} := (BK - 1) \cap K = [-1 - 2a, 1 - 2a] \cap [-a, 1 - a] = [-a, 1 - 2a].$$

Then the endpoints of  $K_{1i}$ ,  $i = 1, 2$  are  $-a, 1 - 2a, 1 - a$ . Let  $E_1 = \{-a, 1 - 2a, 1 - a\}$ . We note that  $2(1 - 2a) = 2 - 4a \neq x + \ell$  for any  $x \in E_1$  and  $\ell \in \mathbb{Z}$ , which implies that  $\{K_{1i}, i = 1, 2\}$  is not an



integral self-affine collection. Thus we need to compute  $C'_2$  and get a new partition  $\{K_{2i}\}_{i=1}^{M_2}$  of the set  $K$ . The endpoints of  $K_{2i}$ ,  $i = 1, 2, \dots, M_2$  belong to the set  $E_2 := \{-a, k - 4a, 1 - 2a, 1 - a, k \in \mathbb{Z}\}$ . Since  $2(k - 4a) = 2k - 8a \neq x_2 + \ell$  for any  $x_2 \in E_2$  and  $\ell \in \mathbb{Z}$ , the collection  $\{K_{2i}\}_{i=1}^{M_2}$  is not an integral self-affine collection. Thus we have to proceed further steps again. Generally, at the  $j^{th}$  step, we get a partition  $\{K_{ji}\}_{i=1}^{M_j}$  of  $K$  and the endpoints of  $K_{ji}$  should be in the set  $E_j := \{-a, k_2 - 2^2a, k_3 - 2^3a, \dots, k_j - 2^ja, 1 - 2a, 1 - a, k_2, \dots, k_j \in \mathbb{Z}\}$ . At the  $j^{th}$  step, there must exist some  $k_j \in \mathbb{Z}$  such that  $k_j - 2^ja$  is one of endpoints of the sets  $K_{ji}$ ,  $i = 1, 2, \dots, M_j$ . However,  $2(k_j - 2^ja) = 2k_j - 2^{j+1}a \neq x + \ell$  for any  $x \in E_j$  and  $\ell \in \mathbb{Z}$ . Therefore,  $\{K_{ji}\}_{i=1}^{M_j}$  is not an integral self-affine collection. Hence, if  $a \notin \mathbb{Q}$ , this process will go on infinitely and for any  $j \in \mathbb{N}$ ,  $\{K_{ji}\}_{i=1}^{M_j}$  is not an integral self-affine collection. This proves that  $K = [-a, 1 - a]$  is not an integral self-affine multi-tile.  $\square$

## References

- [1] T. Flaherty and Y. Wang, *Haar-type multiwavelet bases and self-affine multi-tiles*, *Asian J. Math.*, **3** (1999), pp. 387–400.
- [2] X. Y. Fu and J.-P. Gabardo, *Self-affine scaling sets in  $\mathbb{R}^2$* , *Mem. Amer. Math. Soc.*, (accepted).
- [3] X. Y. Fu and J.-P. Gabardo, *Construction of wavelet sets using integral self-affine multi-tiles*, *J. Fourier Anal. Appl.*, DOI 10.1007/s00041-013-9310-5.
- [4] K. Gröchenig and A. Haas, *Self-similar lattice tilings*, *J. Fourier Anal. Appl.*, **1** (1994), pp. 131–170.
- [5] K. Gröchenig, A. Hass and A. Raugi, *Self-affine tilings with several tiles*, *Appl. Comput. Harmon. Anal.*, **7** (1999), pp. 211–238.
- [6] K. Gröchenig and W. Madych, *Multiresolution analysis. haar bases, and self-similar tilings of  $\mathbb{R}^n$* , *IEEE Trans. Inform. Theory*, **38** (1992), pp. 556–568.
- [7] J. E. Hutchinson, *Fractals and self-similarity*, *Indiana Univ. Math. J.*, **30** (1981), pp. 713–747.
- [8] J. C. Lagarias and Y. Wang, *Self-affine tiles in  $\mathbb{R}^n$* , *Adv. Math.* **121** (1996), pp. 21–49.
- [9] J. C. Lagarias and Y. Wang, *Integral self-affine tiles in  $\mathbb{R}^n$  I. Standard and nonstandard digit sets*, *J. London Math. Soc.*, **54** (1996), pp. 161–179.

- [10] J. C. Lagarias and Y. Wang, *Integral self-affine tiles in  $\mathbb{R}^n$  part II: Lattice tilings*, J. Fourier Anal. Appl., **3** (1997), pp. 83–102.
- [11] J. C. Lagarias and Y. Wang, *Haar-type orthonormal wavelet bases in  $\mathbb{R}^2$* , J. Fourier Anal. Appl., **2** (1995), pp. 1–14.
- [12] J. C. Lagarias and Y. Wang, *Substitution Delone sets*, Discrete Comput. Geom., **29** (2003), pp. 175–209.